

# M4041 Lecture 4

## 7 Cotangent Space and 1-forms

Definition 7.1 Let  $V$  be a finite dimensional vector space. Linear maps

$$\lambda: V \rightarrow \mathbb{R}$$

are called one-forms or co-vectors. The space of all one-forms is denoted by  $V^*$  and referred to as the dual space of  $V$ .

### Proposition 7.2

Suppose  $V$  is an  $n$ -dimensional vector space with basis  $\{e_1, \dots, e_n\}$ . Then  $V^*$  is

an  $n$ -dimensional vector space with basis  $\{\theta^1, \dots, \theta^n\}$  where the  $\theta^j \in V^*$ ,  $j=1, 2, \dots, n$  are uniquely defined by

$$\theta^j(e_i) = \delta_i^j \quad i, j = 1, \dots, n.$$

Terminology The basis  $\{\theta^1, \dots, \theta^n\}$  is known as the dual basis to  $\{e_1, \dots, e_n\}$ .

Proposition 7.3 Let  $V, W$  be finite dimensional vector spaces and suppose  $L: V \rightarrow W$  is a linear map. Then

$$L^* \sigma(v) = \sigma(Lv) \quad \forall v \in V, \sigma \in W^*$$

defines a linear map  $L^*: W \rightarrow V$ , which is called the transpose of  $L$ .

Definition 7.4 The dual space of  $T_p M$

is denoted by  $T_p^* M$ , i.e.

$$T_p^* M = \{ \Theta_p: T_p M \rightarrow \mathbb{R} \mid \Theta \text{ is } \mathbb{R}\text{-linear} \}$$

and called the cotangent space at  $p$ . The set

$$T^* M = \bigcup_{p \in M} T_p^* M$$

is called the cotangent bundle and the map

$$\pi: T^* M \rightarrow M$$

defined by

$$T_p^* M \ni \Theta_p \mapsto \pi(\Theta_p) = p \in M$$

is known as the projection map.

Definition 7.5 Given two manifolds  $M, N$  and a diffeomorphism  $\phi: M \rightarrow N$ , the cotangent lift  $T^*\phi: T^*N \rightarrow T^*M$  of  $\phi$  is defined by

$$T^*\phi(\sigma_q) := \left( T_{\phi(q)}^{-1} \phi \right)^* \sigma_q + \sigma_q \in T_q^*N, q \in N.$$

Proposition 7.6 Suppose  $M, N, P$  are smooth manifolds and  $\phi: M \rightarrow N$  and  $\psi: N \rightarrow P$  are diffeomorphisms. Then

(i) the diagram

$$\begin{array}{ccc} T^*N & \xrightarrow{T^*\phi} & T^*M \\ \pi \downarrow & & \downarrow \\ N & \xrightarrow{\phi^{-1}} & M \end{array}$$

- commutes, and

$$(ii) \quad T^*(\psi \circ \phi) = T^*\phi \circ T^*\psi.$$

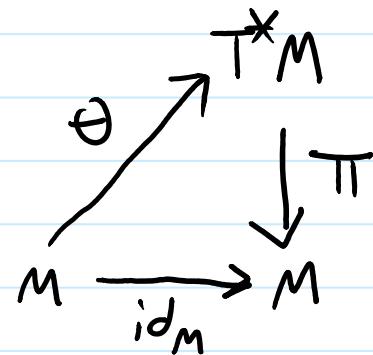
Proof Homework.

The cotangent bundle  $T^*M$  is a smooth manifold (<sup>HW: Show this</sup>), and a smooth map

$$\theta: M \rightarrow T^*M$$

satisfying

$$\pi \circ \theta = \text{id}_M$$



is known as a 1-form. The space

of all one forms is denoted by  $\mathcal{E}^*(M)$  and it can be shown to be a  $C^\infty(M)$ -module.

Remark 7.7 Each  $\theta \in \mathcal{E}^*(M)$  defines

a linear map :

$$\theta: \mathcal{E}(M) \rightarrow C^\infty(M): X \mapsto \theta(X)$$

Definition 7.8 The exterior derivative is the map

$$d: C^\infty(M) \rightarrow \mathcal{E}^*(M)$$

defined by

$$(dS)_p(X_p) = X_p(S) \quad \forall p \in M, X \in \mathcal{E}(M).$$

Suppose that  $(U, \varphi = (\varphi^1, \dots, \varphi^n))$   
is a chart on  $M$  that contains  $p \in M$ .

Then we can define a basis

$$\{d\varphi^1|_p, \dots, d\varphi^n|_p\} \text{ of } T_p^*M$$

by defining

$$d\varphi^i|_p\left(\frac{\partial}{\partial \varphi^j}|_p\right) = \delta_j^i$$

\* Note: this formula defines  $d\varphi^i$ . It should not yet be interpreted as the exterior derivative of  $\varphi^i$ .

The basis  $\{d\varphi^1|_p, \dots, d\varphi^n|_p\}$  of  $T_p^*M$

defined this way is known as the basis dual to  $\{\frac{\partial}{\partial \varphi^1}|_p, \dots, \frac{\partial}{\partial \varphi^n}|_p\}$ , or just the dual basis.

We also note that the

$$d\varphi^i : U \subset M \rightarrow T^*U : p \mapsto d\varphi^i|_p$$

define 1-forms on  $U$ .

### Exercise 7.9

Given  $C^\infty(M)$  and a chart  $(U, \varphi = (\varphi^1, \dots, \varphi^n))$  on  $M$ . Show that

$$df(p) = \frac{\partial f}{\partial \varphi^i} d\varphi^i$$

This is not interpreted here as the exterior derivative of  $\varphi^i$ , but via the previous definition as the dual basis.

Remark: This formula justifies the notation  $d\varphi^i$  for the dual basis, since if we choose  $f$  to be the coordinate function, that is

$$f = \varphi^j$$

then

$$d\varphi^i = \cancel{\frac{\partial \varphi^i}{\partial \varphi^j} df_j} \stackrel{\varphi^i}{\rightarrow} d\varphi^i = df^i$$

exterior derivative of  $\varphi^i$     dual basis interpretation

Lemma 7.10 The exterior derivative  $d : C^\infty(M) \rightarrow \Omega^*(M)$  satisfies the following properties:

(i)  $d$  is  $\mathbb{R}$ -linear,

$$(ii) \quad d(fg) = f dg + g df$$

for all  $f, g \in C^\infty(M)$ , and

$$(iii) \quad d(hof) = h'(f) df$$

for all  $f \in C^\infty(M)$  and  $h \in C^\infty(\mathbb{R})$ .

## Definition 7.11

Given a map  $\phi \in C^\infty(M, N)$  and a 1-form  $\Theta \in \mathcal{X}^*(N)$ , the pull-back of  $\Theta$  by  $\phi$ , denoted  $\phi^*\Theta$ , is the (unique) 1-form  $\phi^*\Theta \in \mathcal{X}^*(M)$

defined by

$$(\phi^*\Theta)(p)(v_p) = \Theta(\phi(p)) (T_p\phi \cdot v_p) \quad \forall p \in M, v_p \in T_p M.$$

## Lemma 7.12

Suppose  $\phi \in C^\infty(M, N)$  and  $f \in C^\infty(N)$ .  
Then

$$\phi^*(df) = d(\phi^*f)$$

where  $\phi^*f = f \circ \phi$ .

Proof Homework.

## Local Coordinates on $T^*M$

Given a coordinate chart  $(U, \varphi)$  on a manifold  $M$ , we have that

$$\overline{T}^*\varphi^{-1}: \overline{T}^*U \longrightarrow \overline{T}^*\varphi(U)$$

Just as for the tangent space, there is a natural identification

$$\overline{T}^*\varphi(U) \cong \varphi(U) \times \mathbb{R}^n$$

given by

$$\overline{T}^*\varphi(U) \ni \sigma_x = \sigma_i dx^i \mapsto (x, \sigma) \in \varphi(U) \times \mathbb{R}^n$$

which allows us to view  $\overline{T}^*\varphi^{-1}$  as a map

$$\overline{T}^*\varphi^{-1}: \overline{T}^*U \longrightarrow \varphi(U) \times \mathbb{R}^n$$

and to view a dual vector locally as a pair

$$(x, \sigma) = (x^i, \sigma_j)$$

of  $n$ -dimensional vectors. Moreover, if

$$\mathcal{A} = \bigcup_{\alpha \in I} \{(U_\alpha, \varphi_\alpha)\}$$

is an atlas for  $M$ , then

$$T^*\mathcal{A} = \{(T\psi_\alpha, T\psi_\alpha^{*-1})\}$$

determines an atlas for  $T^*M$  and shows that  $T^*\mathcal{A}$  is a smooth manifold of dimension  $2n$ , and in fact, a rank  $n$  vector bundle.

### Exercise 7.13

Suppose that  $U$  and  $V$  are open sets in  $\mathbb{R}^n$  and  $\phi: U \rightarrow V$  is a diffeomorphism. Show that

$$T^*\phi(x, \sigma_i) = (\bar{\phi}(x), \partial_i \bar{\phi}^j(\bar{\phi}(x)) \bar{\sigma}_j)$$

for all  $(x, \sigma_i) \in V \times \mathbb{R}^n \cong T^*V$ , where

in components

$$\bar{\phi}(x) = (\phi'(x), \dots, \phi^n(x))$$

### Local 1-form (fields)

Given a 1-form  $\Theta \in \mathcal{X}^*(M)$  and a chart  $(U, \varphi)$  on  $M$ , we define the local version

$\Theta^\varphi \in \mathcal{X}^*(\varphi(U))$  by

$$\Theta^\varphi := (\bar{\varphi})^* \Theta.$$

Since  $\Theta$  satisfies  $\pi \circ \Theta = \text{id}_M$ , it follows that the local expression must be of the form

$$\Theta^\varphi(x) = (x, \hat{\Theta}^\varphi(x)) \quad \forall x \in \varphi(U)$$

where

$$\hat{\Theta}^\varphi \in C^\infty(\varphi(U), \mathbb{R}^n).$$

Just as in the case for vector fields, it is common to refer to the vector component

$$\hat{\Theta}^\varphi(x) = (\hat{\Theta}_1^\varphi(x), \dots, \hat{\Theta}_n^\varphi(x))$$

as the local representation of  $\Theta$  in the chart  $(U, \varphi)$ . It is even common to write this as

$$\hat{\Theta}_i^\varphi(x).$$

### Exercise 7.14

Suppose that  $(U, \varphi = (\varphi^1, \dots, \varphi^n))$  is a chart on  $M$ ,  $\Theta \in \mathcal{X}^*(M)$  and

$$\Theta = \Theta_i^j d\varphi^i \quad \text{in } U$$

is the expansion of  $\Theta$  in the dual basis  $\{d\varphi^1, \dots, d\varphi^n\}$ .

Next let

$$\Theta^\varphi(x) := (\varphi^{-1})^* \Theta(x) = (x, \hat{\Theta}^\varphi(x))$$

be the local representation of  $\Theta$ .

Show that

$$\Theta_i^{\circ\varphi} = \hat{\Theta}_i^\varphi \text{ in } \varphi(U)$$

### Exercise 7.15

Suppose that  $(U, \varphi = (\varphi^1, \dots, \varphi^n))$  and  $(V, \psi = (\psi^1, \dots, \psi^n))$  are two charts on  $M$  that overlap smoothly, and that

$$\Theta_i^\varphi(x) \quad x \in \varphi(U)$$

$$\Theta_j^\psi(x) \quad x \in \psi(V)$$

are the local representations of  $\Theta$  in the charts  $(U, \varphi)$  and  $(V, \psi)$ , respectively.

Show that

$$\Theta_i^\varphi(x) = \sum_j \tilde{\gamma}_{\varphi\psi}^j(x) \Theta_j^\psi \tilde{\gamma}_{\psi\varphi}^i(x) \quad \forall x \in \varphi(U \cap V)$$

where  $\tilde{\gamma}_{\varphi\psi} = \psi \circ \varphi^{-1}$

## 8. Tensors

Definition 8.1 Let  $V$  be real, finite dimensional vector space. A tensor of type  $(r,s)$  is a multilinear function

$$A : \underbrace{V^* \times \cdots \times V^*}_{r\text{-times}} \times \underbrace{V \times \cdots \times V}_{s\text{-times}} \rightarrow \mathbb{R}$$

The set of  $(r,s)$  tensors is denoted by  $T_s^r(V)$  and is a real, finite dimensional vector space.

Given  $A \in T_s^r(V)$  and  $B \in T_q^p(V)$ ,

the tensor product of  $A$  and  $B$  is the element

$$A \otimes B \in T_{s+q}^{r+p}(V)$$

defined by

$$A \otimes B(\theta^1, \dots, \theta^{r+p}, e_1, \dots, e_{s+q})$$

$$:= A(\theta^1, \dots, \theta^r, e_1, \dots, e_s) B(\theta^{r+1}, \dots, \theta^{r+p}, e_{s+1}, \dots, e_{s+q})$$

It is common to call  $r$  the contravariant rank and  $s$  the covariant rank of a  $(r,s)$  type tensor.

## Examples 8.2

(i)  $V^* = T_1^0(V)$  by definition.

(ii)  $V \cong T_0^1(V)$  because  $(V^*)^* \cong V$ .

(iii)  $\overline{T}_2^0(V)$  is the space of bilinear forms on  $V$ .

Lemma 8.3 Suppose  $V$  is a vector space,  $\{e_1, \dots, e_n\}$  is a basis for  $V$ , and  $\{\theta^1, \dots, \theta^n\}$  is the dual basis (i.e.  $\theta^i(e_j) = \delta_{ij}$ ). Then

$\{e_{i_1} \otimes \dots \otimes e_{i_r} \otimes \theta^{j_1} \otimes \dots \otimes \theta^{j_s} \mid i_1, j_1 \in \{1, \dots, n\}, l=1, \dots, r, m=1, \dots, s\}$

is a basis of  $T_s^r(V)$  and  $\dim T_s^r = n^{r+s}$ .

From the above lemma, we see that we can expand a tensor  $A \in T_s^r(V)$  in terms of the given basis as follows:

$$A = A_{j_1, \dots, j_s}^{i_1, \dots, i_r} e_{i_1} \otimes \dots \otimes e_{i_r} \otimes \theta^{j_1} \otimes \dots \otimes \theta^{j_s}$$

Here, the expansion coefficients  $A_{j_1, \dots, j_s}^{i_1, \dots, i_r}$  are called the components of  $A$  with respect to the basis.

Furthermore, if

$$X_l = X_l^i e_i \quad l=1, \dots, r$$

$$\varphi^m = \varphi_i^m \theta^i \quad m=1, \dots, s$$

Then

$$A(\varphi_1^1, \dots, \varphi_s^m, X_1, \dots, X_r) = A_{j_1, \dots, j_s}^{i_1, \dots, i_r} \varphi_1^{i_1} \dots \varphi_s^{i_s} X_1^{j_1} \dots X_r^{j_s}$$

Example 8.4 Let  $g \in T_2^0(V)$  and  $X, Y \in V$   
so that

$$X = X^i e_i, \quad Y = Y^i e_i$$

and

$$g = g_{ij} \theta^i \otimes \theta^j$$

Then

$$g(X, Y) = g_{ij} X^i Y^j$$

Exercise 8.5

Show that  $T_1^1(V) \cong L(V)$  where  
 $L(V)$  is the set of linear maps on  $V$ .

Hint: Show that the map

$$\varphi: L(V) \rightarrow T_1^1(V)$$

defined by

$$\varphi(A)(\theta, x) := \theta(A(x)) \quad \forall A \in L(V), X \in V, \theta \in V^*$$

is an isomorphism.

Definition 8.6 Let  $M$  be a manifold and  $p \in M$ . Then a tensor of type  $(r,s)$  at  $p$  is an element of

$$T_s^r(T_p M).$$

The

$$\overline{T}_s^r M := \bigcup_{p \in M} T_s^r(T_p M)$$

is called the tensor bundle of type  $(r,s)$

and it can be shown that  $\overline{T}_s^r(M)$  is a smooth manifold of dimension  $n + n^{r+s}$ , and the projection map

$$\pi: \overline{T}_s^r M \rightarrow M$$

defined by

$$\overline{T}_s^r(T_p M) \ni A_p \mapsto \pi(A_p) := p \in M.$$

is smooth.

A tensor field of type  $(r,s)$  is a smooth map

$$A: M \longrightarrow T_s^r M$$

satisfying

$$\bar{\pi} \circ A = \text{id}_M.$$

$$\begin{array}{ccc} & & T_s^r(M) \\ & A \nearrow & \downarrow \pi \\ M & \longrightarrow & M \\ & & \text{id}_M \end{array}$$

The space of tensor field of type  $(r,s)$  is denoted by  $\Gamma(T_s^r M)$ .

### Definition 8.7

Given a diffeomorphism  $\Phi \in C^\infty(M, N)$ , the map

$$T_s^r \Phi : T_s^r M \rightarrow T_s^r N$$

defined by

$$\begin{aligned} T_s^r \Phi(A)(\sigma^1, \dots, \sigma^r, v_1, \dots, v_s) \\ = A((T_p \bar{\Phi})^* \sigma^1, \dots, (T_p \bar{\Phi})^* \sigma^r, \bar{T}_{\Phi(p)} \bar{\Phi}^{-1} v_1, \dots, \bar{T}_{\Phi(p)} \bar{\Phi}^{-1} v_s) \\ \forall A \in T_s^r T_p M ; v_1, \dots, v_s \in T_{\Phi(p)}^* N ; \sigma_1, \dots, \sigma_r \in T_{\Phi(p)}^* N \end{aligned}$$

is called the tensor bundle lift of  $\Phi$ .

## Exercise 8.8

Given diffeomorphisms  $\Phi \in C^\infty(M, N)$  and  $\Psi \in C^\infty(N, P)$ , show that

$$T_S^r(\bar{\Psi} \circ \bar{\Phi}) = T_S^r \bar{\Psi} \circ T_S^r \bar{\Phi}$$

Local Coordinates on  $T_S^r M$

Given a coordinate chart  $(U, \varphi)$  on a manifold  $M$ , we have that

$$T_S^r \varphi : T_S^r U \longrightarrow T_S^r \varphi(U)$$

Just as for the tangent space, there is a natural identification

$$T_S^r \varphi(U) \cong \varphi(U) \times \mathbb{R}^{n^{r+s}}$$

given by

$$T_S^r T_x^r U \ni A = A_{j_1 \dots j_s}^{i_1 \dots i_r} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s} \mapsto (x, A_{j_1 \dots j_s}^{i_1 \dots i_r})$$

which allows us to view  $T_S^r \varphi$  as a map

$$T_S^r \varphi : T_S^r U \longrightarrow \varphi(U) \times \mathbb{R}^{n^{r+s}}$$

and to view a tensor locally as a pair

$$(x, A) = (x^i, A_{j_1 \dots j_s}^{i_1 \dots i_r})$$

of vectors of dimension  $n$  and  $n^{r+s}$ , respectively. Moreover, if

$$\mathcal{A} = \bigcup_{\alpha \in I} \{(U_\alpha, \varphi_\alpha)\}$$

is an atlas for  $M$ , then

$$T_s^r \mathcal{A} = \bigcup_{\alpha \in I} \{(T_s^r U_\alpha, T_s^r \varphi_\alpha)\}$$

is an atlas for  $T_s^r M$

### Remark 8.9

We also note that if  $A \in \Gamma(T_s^r(M))$ , then, in local coordinates  $(U, \varphi = (\varphi^1, \dots, \varphi^n))$ ,  $A = A^{i_1 \dots i_r}_{j_1 \dots j_s} \frac{\partial}{\partial \varphi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial \varphi^{i_r}} \otimes d\varphi^{j_1} \otimes \dots \otimes d\varphi^{j_s}$ . and the components  $A^{i_1 \dots i_r}_{j_1 \dots j_s}$  are smooth.

Remark 8.10 An tensor field  $A \in \Gamma(T_s^r M)$  induces an  $C^\infty(M)$ -multilinear map

$$A: \underbrace{\mathcal{X}(M) \times \dots \times \mathcal{X}(M)}_{r \text{ times}} \times \underbrace{\mathcal{X}(M) \times \dots \times \mathcal{X}(M)}_{s \text{ times}} \rightarrow \mathcal{J}(M)$$

$$\text{by } A(\theta^1, \dots, \theta^r, X_1, \dots, X_s)(p)$$

$$= A(p)(\theta^1(p), \dots, \theta^r(p), X_1(p), \dots, X_s(p)) \quad \forall p \in M$$

Terminology : Tensor fields from  $\Gamma(T_0^r M)$  are called contravariant while tensors fields from  $\Gamma(T_s^r M)$  are called covariant.

Definition 8.11 Given a smooth map

$\phi: M \rightarrow N$ , we define the pull back of a covariant tensor field  $A \in \Gamma(T_s^r N)$

$$(\phi^* A)(p)(x_1, \dots, x_s) = A(\phi(p))(T_p \phi x_1, \dots, T_p \phi x_s)$$

for all  $p \in M$  and  $x_1, \dots, x_s \in T_p M$ .

If  $\phi: M \rightarrow N$  is a diffeomorphism, we define the pull back of a tensor field  $B \in T_s^r(N)$  by

$$(\phi^* B)(p)(\partial'_1, \dots, \partial'_r, x_1, \dots, x_s)$$

$$= B(\phi(p))\left((T_{\phi(p)} \bar{\phi}^{-1})^* \partial'_1, \dots, (T_{\phi(p)} \bar{\phi}^{-1})^* \partial'_r, T_p \phi x_1, \dots, T_p \phi x_s\right)$$

$$\text{for all } \partial'_1, \dots, \partial'_r \in T_p^* M \text{ and } x_1, \dots, x_s \in T_p M.$$

Finally, for  $C \in \Gamma(T_S^r M)$ , we define the push forward of  $C$  by  $\phi$

$$\phi_* C = (\phi^{-1})^* C$$

Given a tensor

$$A \in T_S^r(T_p M)$$

we can expand it in the coordinate basis to get

$$A = A_{j_1 \dots j_s}^{i_1 \dots i_r} \frac{\partial}{\partial \varphi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial \varphi^{i_r}} \otimes d\varphi^{j_1} \otimes \dots \otimes d\varphi^{j_s}$$

We then define the contraction of  $A$  on the  $i_e^{\text{th}}$  contravariant and  $j_m^{\text{th}}$  covariant index by

$$C(A) = C(A)_{j_1 \dots j_{s-1}}^{i_1 \dots i_{r-1}} \frac{\partial}{\partial \varphi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial \varphi^{i_{r-1}}} \otimes d\varphi^{j_1} \otimes \dots \otimes d\varphi^{j_{s-1}}$$

where

$$C(A)_{j_1 \dots j_{s-1}}^{i_1 \dots i_{r-1}} = A_{j_1 \dots j_{m-1} \dots j_{s-1}}^{i_1 \dots i_{r-1} + i_m \dots i_{r-1}}$$

For example, given  $A = A^i_j \frac{\partial}{\partial x^i} \otimes dx^j$   
 in  $T^r_s M$ , we have that

$$c(A) = A^i_i$$

### Exercise 8.12

Show that contraction defines a map

$$c : T^r_s(M) \longrightarrow T^{r-1}_{s-1}(M)$$

### Exercise 8.13

Suppose that  $(U, \varphi = (\varphi^1, \dots, \varphi^n))$  is a chart  
 on  $M$ ,  $A \in \Gamma(T^r_s M)$  and

$$A = A^i_{j_1 \dots j_r} \frac{\partial}{\partial \varphi^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial \varphi^{j_r}} \otimes d\varphi^{j_1} \otimes \dots \otimes d\varphi^{j_r}$$

is the expansion of  $A$  in the coordinate's basis.

Next, let

$$A_\varphi(x) := (\varphi)_* A(x) = (x, \hat{A}_\varphi(x))$$

be the local representation of  $A$ .

Show that

$$A^i_{j_1 \dots j_r} \circ \varphi = (\hat{A}_\varphi)_{j_1 \dots j_r}^{i_1 \dots i_r} \text{ in } \varphi(U).$$

## Exercise 8.14

Suppose that  $(U, \varphi = (\varphi^1, \dots, \varphi^n))$  and  $(V, \psi = (\psi^1, \dots, \psi^n))$  are two charts on  $M$  that overlap smoothly and that

$$A_{\varphi_{U_1 \cup U_2}}^{i_1 \dots i_r}(x) \quad x \in \varphi(U) \text{ and } A_{\psi_{V_1 \cup V_2}}^{i_1 \dots i_r}(x) \quad x \in \psi(V)$$

are the local representatives of  $A$  in the charts  $(U, \varphi)$  and  $(V, \psi)$  respectively.

Show that

$$A_{\varphi_{U_1 \cup U_2}}^{i_1 \dots i_r} = \partial \tilde{\zeta}^{l_1} \dots \partial \tilde{\zeta}^{l_s} \partial (\tilde{\zeta}^{-1})^{i_1} \dots \partial (\tilde{\zeta}^{-1})^{i_r} A_{\psi_{V_1 \cup V_2}}^{k_1 \dots k_r} \circ \tilde{\zeta} \text{ in } \varphi(U \cap V)$$

where  $\tilde{\zeta} = \varphi \circ \psi^{-1}$ .

Definition 8.15 A metric on  $M$  is an element  $g \in \Gamma(T_2^0(M))$  that satisfies the following properties:

(i)  $g$  is symmetric, that is

$$g(p)(v_p, w_p) = g(p)(w_p, v_p)$$

for all  $p \in M$  and  $v_p, w_p \in T_p M$ ,

(ii)  $g$  is non-degenerate, that is

$$g(p)(v_p, w_p) = 0 \quad \nabla w_p \in T_p M$$

implies that  $v_p = 0$ .

By the Principle Axis Theorem, around any  $p \in M$  we can introduce a local coordinate-system so that

$$g(p) = g_{ij}(p) dy^i \otimes dy^j \quad i,j=1,\dots,n$$

where

$$(g_{ij}(p)) = \text{diag} \left( \underbrace{-1, \dots, -1}_{s \text{ times}}, \underbrace{+1, \dots, +1}_{n-s \text{ times}} \right)$$

If  $s=0$  for every  $p \in M$ , then  $g$  is called a Riemannian metric.

If  $s=1$  or  $s=n-1$  for every  $p \in M$ , then  $g$  is called a Lorentzian metric.

### Proposition 8.16

Every  $C^\infty(M)$ -multilinear map

$$A : \underbrace{\mathcal{X}^*(M) \times \dots \times \mathcal{X}^*(M)}_r \times \underbrace{\mathcal{X}(M) \times \dots \times \mathcal{X}(M)}_s \rightarrow C^\infty(M)$$

defines a tensor field  $\tilde{A} \in T_s^r(M)$  according to the prescription

for  $p \in M$ ,  $\sigma^1, \dots, \sigma^r \in T_p^*M$ ,  $v_1, \dots, v_s \in T_p M$ ,

$$\tilde{A}(p)(\sigma^1, \dots, \sigma^r, v_1, \dots, v_s) := [A(\theta^1, \dots, \theta^r, x_1, \dots, x_s)](p),$$

where  $\theta^1, \dots, \theta^r \in \mathcal{X}^*(M)$  and  $x_1, \dots, x_s \in \mathcal{X}(M)$

are any collection of differential 1-forms  
and vector fields satisfying

$$\theta^i(p) = \sigma^i \quad i=1, \dots, r$$

and

$$x_i(p) = v_i \quad i=1, \dots, s.$$

## Proof Homework